AN INDEFINITE SUPERLINEAR ELLIPTIC EQUATION WITH A NONLINEAR BOUNDARY CONDITION OF SUBLINEAR TYPE (Shapes and other properties of solutions of PDEs)

AUTHOR(S):

Quoirin, Humberto Ramos; Umezu, Kenichiro

CITATION:

Quoirin, Humberto Ramos ...[et al]. AN INDEFINITE SUPERLINEAR ELLIPTIC EQUATION WITH A NONLINEAR BOUNDARY CONDITION OF SUBLINEAR TYPE (Shapes and other properties of solutions of PDEs). 数理解析研究所講究録 2016, 2006: 68-88

ISSUE DATE:

2016-11

URL:

http://hdl.handle.net/2433/231534

数理解析研究所講究録
第2006巻 2016年 68-88

AN INDEFINITE SUPERLINEAR ELLIPTIC EQUATION WITH A NONLINEAR BOUNDARY CONDITION OF SUBLINEAR TYPE

HUMBERTO RAMOS QUOIRIN AND KENICHIRO UMEZU

ABSTRACT. We investigate an indefinite superlinear elliptic equation coupled with a sublinear Neumann boundary condition depending on a positive parameter λ. We establish a global multiplicity result for positive solutions of this concave-convex problem in the spirit of Ambrosetti-Brezis-Cerami and obtain their asymptotic profiles as $\lambda\rightarrow 0^{+}$. Furthermore, we discuss the existence of a global subcontinuum of positive solutions bifurcating from the trivial solutions. Our arguments are based on a bifurcation analysis, a comparison principle, variational techniques, and a topological method.

1. INTRODUCTION AND STATEMENTS OF MAIN RESULTS

Let Ω be a bounded domain of $\mathbb{R}^{N}$ $(N\geq 2)$ with smooth boundary $\partial\Omega$. In this paper we consider the following nonlinear elliptic problem

$$\begin{cases} -\Delta u=a(x)|u|^{p-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}}=\lambda|u|^{q-2}u & \text{on } \partial\Omega, \end{cases} \tag{P_{λ}}$$

where

- $\Delta=\sum_{j=1}^{N}\frac{\partial^{2}}{\partial x_{j}^{2}}$ is the usual Laplacian in $\mathbb{R}^{N}$,
- $\lambda>0$,
- $1<q<2<p<\infty$,
- $a\in C^{\alpha}(\overline{\Omega})$ with $\alpha\in(0,1)$,
- $\mathbf{n}$ is the unit outer normal to the boundary $\partial\Omega$.

A function $u\in X:=H^{1}(\Omega)$ is said to be a *weak solution* of (P_{λ}) if it satisfies

$$\int_{\Omega}\nabla u\nabla w-\int_{\Omega}a|u|^{p-2}uw-\lambda\int_{\partial\Omega}|u|^{q-2}uw=0, \quad \forall w\in X.$$

A weak solution u of (P_{λ}) is said to be *nontrivial and non-negative* if it satisfies $u\geq 0$ and $u\not\equiv 0$. Under the condition

$$p\leq 2^{*}=\frac{2N}{N-2} \quad \text{if } N>2, \tag{1.1}$$

we shall prove that such solutions are strictly positive on $\overline{\Omega}$ (Proposition 2.1) and belong to $C^{2+\theta}(\overline{\Omega})$ for some $\theta\in(0,1)$ (Remark 2.2). To this end, we use the weak maximum principle [12] to deduce that any nontrivial non-negative weak solution u of (P_{λ}) is strictly positive in Ω. In addition, by making good use of a comparison principle [16, Proposition A.1], we shall prove that u is positive on the whole of $\overline{\Omega}$. Finally, a bootstrap argument will provide $u\in C^{2+\theta}(\overline{\Omega})$ for some $\theta\in(0,1)$, so that u is a (*classical*) *positive solution*. Note that

2010 *Mathematics Subject Classification.* 35J25, 35J61, 35J20, 35B09, 35B32.
Key words and phrases. Semilinear elliptic problem, Concave-convex nonlinearity, Nonlinear boundary condition, Positive solution, Bifurcation, Super and subsolutions, Nehari manifold.

The first author was supported by the FONDECYT grant 11121567.
The second author was supported by JSPS KAKENHI Grant Number 15K04945.

the standard boundary point lemma (as in [14]) can not be applied directly to nontrivial non-negative weak solutions of (P_λ).

The purpose of this paper is to study existence, non-existence, and multiplicity of positive solutions of (P_λ), as well as their asymptotic properties as the parameter λ approaches 0. It is promptly seen that (P_λ) has no positive solution if $a \geq 0$. More precisely, we shall see that (P_λ) has a positive solution only if $\int_\Omega a < 0$ (cf. Proposition 2.3). This condition is known to be necessary for the existence of positive solutions of problems with Neumann boundary conditions at least since the work of Bandle-Pozio-Tesei [3]. In this paper we focus on the case where a changes sign.

In view of the condition $1 < q < 2 < p$, we note that if a changes sign then (P_λ) belongs to the class of concave-convex type problems with nonlinear boundary conditions. The main reference on concave-convex type problems is the work of Ambrosetti-Brezis-Cerami [2], which deals with

$$\begin{cases} -\Delta u = \lambda |u|^{q-2}u + |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $1 < q < 2 < p$. Under the condition (1.1) the authors proved a *global multiplicity result*, namely, the existence of some $\Lambda > 0$ such that (1.2) has at least two positive solutions for $\lambda \in (0, \Lambda)$, at least one positive solution for $\lambda = \Lambda$, and no positive solution for $\lambda > \Lambda$. In addition, they analysed the asymptotic behavior of the solutions as $\lambda \to 0^+$. Tarfulea [21] considered a similar problem with an indefinite weight and a Neumann boundary condition, namely,

$$\begin{cases} -\Delta u = \lambda |u|^{q-2}u + a(x)|u|^{p-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $a \in C(\overline{\Omega})$. He proved that $\int_\Omega a < 0$ is a necessary and sufficient condition for the existence of a positive solution of (1.3). Making use of the sub-supersolutions technique, he has also shown the existence of $\Lambda > 0$ such that problem (1.3) has at least one positive solution for $\lambda < \Lambda$ which converges to 0 in $L^\infty(\Omega)$ as $\lambda \to 0^+$, and no positive solution for $\lambda > \Lambda$. Garcia-Azorero, Peral, and Rossi [10] have considered the problem

$$\begin{cases} -\Delta u + u = |u|^{p-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = \lambda |u|^{q-2}u & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

By means of a variational approach, they proved that if $1 < q < 2 < p$ and $p < 2^*$ when $N > 2$, then there exists $\Lambda_0 > 0$ such that (1.4) has infinitely many nontrivial weak solutions for $0 < \lambda < \Lambda$. Moreover, they have also proved that if $1 < q < 2$ and $p = 2^*$ when $N > 2$ then there exists $\Lambda_1 > 0$ such that (1.4) has at least two positive solutions for $\lambda < \Lambda_1$, at least one positive solution for $\lambda = \Lambda_1$, and no positive solution for $\lambda > \Lambda_1$.

When a changes sign we shall prove a global multiplicity result in the style of Ambrosetti-Brezis-Cerami result. However, in doing so we shall encounter some particular difficulties. First of all, the obtention of a first solution by the sub-supersolution method seems difficult since the existence of a strict supersolution of (P_λ) for $\lambda > 0$ small is not evident at all. As a matter of fact, in [21] the author shows that this is a rather delicate issue. Another difficulty in this case is related to the variational structure: note that unlike in problems with Dirichlet boundary conditions, the left-hand side of (P_λ) lacks coercivity, since the term $\int_\Omega |\nabla u|^2$ does not correspond to $\|u\|^2$ in X. This sort of problems has been considered in [15, 16] for other kinds of nonlinearities and we shall use a similar approach here to prove existence results for (P_λ). This approach is based on the Nehari manifold method, which is known to be useful when dealing with elliptic problems with powerlike nonlinearities and sign-changing weights. Brown and Wu [5] used this method to deal with the problem

$$\begin{cases} -\Delta u = \lambda m(x)|u|^{q-2}u + a(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where m, a are smooth functions which are positive somewhere in Ω. We refer also to Brown [4] for a combination of sublinear and linear terms and to Wu [23] for a problem with a nonlinear boundary condition.

Whenever $\int_\Omega a < 0$ we set

$$c^* = \left(\frac{|\partial\Omega|}{-\int_\Omega a} \right)^{\frac{1}{p-q}}. \tag{1.6}$$

We also set

$$\overline{\lambda} = \sup\{\lambda > 0 : (P_\lambda) \text{ has a positive solution}\}.$$

Let us recall that a positive solution u of (P_λ) is said to be *asymptotically stable* (respect. *unstable*) if $\gamma_1(\lambda, u) > 0$ (respect. < 0), where $\gamma_1(\lambda, u)$ is the smallest eigenvalue of the linearized eigenvalue problem at u, namely,

$$\begin{cases} -\Delta\phi = (p-1)a(x)u^{p-2}\phi + \gamma\phi & \text{in } \Omega, \\ \frac{\partial\phi}{\partial\mathbf{n}} = \lambda(q-1)u^{q-2}\phi + \gamma\phi & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

In addition, u is said *weakly stable* if $\gamma_1(\lambda, u) \geq 0$.

We state now our main result:

Theorem 1.1.

(1) (P_λ) has a positive solution for $\lambda > 0$ sufficiently small if

$$\int_\Omega a < 0. \tag{1.8}$$

Conversely, if (P_λ) has a positive solution for some $\lambda > 0$ then (1.8) is satisfied.

(2) Assume (1.8). Then the following assertions hold:

(a) $0 < \overline{\lambda} \leq \infty$ and (P_λ) has a minimal positive solution $\underline{u}_\lambda$ for $\lambda \in (0, \overline{\lambda})$, i.e. any positive solution u of (P_λ) satisfies $\underline{u}_\lambda \leq u$ in $\overline{\Omega}$. Furthermore $\underline{u}_\lambda$ has the following properties:

(i) $\lambda \mapsto \underline{u}_\lambda(x)$ is strictly increasing in $(0, \overline{\lambda})$.

(ii) $\underline{u}_\lambda$ is asymptotically stable for every $\lambda \in (0, \overline{\lambda})$.

(iii) $\lambda \mapsto \underline{u}_\lambda$ is C^∞ from $(0, \overline{\lambda})$ to $C^{2+\alpha}(\overline{\Omega})$.

(iv) $\underline{u}_\lambda \to 0$ and $\lambda^{-\frac{1}{p-q}}\underline{u}_\lambda \to c^*$ in $C^{2+\alpha}(\overline{\Omega})$ as $\lambda \to 0^+$.

(b) Assume (1.1). If $\overline{\lambda} < \infty$ then (P_λ) has a minimal positive solution $\underline{u}_{\overline{\lambda}}$ for $\lambda = \overline{\lambda}$. Moreover the solution set around $(\overline{\lambda}, \underline{u}_{\overline{\lambda}})$ consists of a C^∞-curve $(\lambda(s), u(s)) \in \mathbb{R} \times C^{2+\alpha}(\overline{\Omega})$ of positive solutions, which is parametrized by $s \in (-\varepsilon, \varepsilon)$, for some $\varepsilon > 0$, and satisfies $(\lambda(0), u(0)) = (\overline{\lambda}, \underline{u}_{\overline{\lambda}})$, $\lambda'(0) = 0$, $\lambda''(0) < 0$, and $u(s) = \underline{u}_{\overline{\lambda}} + s\phi_1 + z(s)$, where ϕ_1 is a positive eigenfunction associated to the smallest eigenvalue $\gamma_1(\overline{\lambda}, \underline{u}_{\overline{\lambda}})$ of (1.7), and $z(0) = z'(0) = 0$. Finally, the lower branch $(\lambda(s), u(s))$, $s \in (-\varepsilon, 0)$, is asymptotically stable, whereas the upper branch $(\lambda(s), u(s))$, $s \in (0, \varepsilon)$, is unstable.

(c) Assume $p < 2^*$ if $N > 2$. Then the set of positive solutions of (P_λ) for $\lambda > 0$ around $(\lambda, u) = (0,0)$ in $\mathbb{R} \times X$ consists of $\{(\lambda, \underline{u}_\lambda)\}$.

(d) Bifurcation from zero of (P_λ) never occurs at any $\lambda > 0$, i.e. there is no sequence (λ_n, u_n) of positive solutions of (P_λ) such that $u_n \to 0$ in $C(\overline{\Omega})$ and $\lambda_n \to \lambda^* > 0$.

(e) (P_λ) has at most one weakly stable positive solution.

Remark 1.2.

(1) Under conditions (1.8) and (1.1), by the left-continuity of $\underline{u}_\lambda$ [1, Theorem 20.3], we infer that $(\lambda(s), u(s))$, $s \in (-\varepsilon, 0)$, in Theorem 1.1(2)(b) represents minimal positive solutions. In particular, the mapping $\lambda \mapsto \underline{u}_\lambda$ is continuous from $(0, \overline{\lambda}]$ into $C(\overline{\Omega})$.

(2) Under (1.1) the minimal positive solution $\underline{u}_{\overline{\lambda}}$ obtained for $\lambda = \overline{\lambda}$ satisfies in addition $\gamma_1(\overline{\lambda}, \underline{u}_{\overline{\lambda}}) = 0$.

(3) In accordance with Theorem 1.1, if $\overline{\lambda} < \infty$ then the set of bifurcating positive solutions at $(0,0)$ is represented in Figure 1.

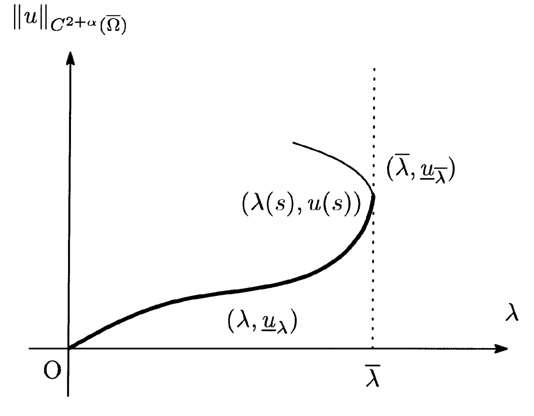

FIGURE 1. A smooth positive solution curve when $\overline{\lambda} < \infty$.

Theorem 1.3. *Assume that a changes sign and* (1.8) *is satisfied. Then the following assertions hold:*

(1) If $a > 0$ on $\partial\Omega$ then $\overline{\lambda} < \infty$.

(2) Assume in addition $p < \frac{2N}{N-2}$ if $N > 2$. Then (P_λ) has a second positive solution $u_{2,\lambda}$ satisfying $\underline{u}_\lambda < u_{2,\lambda}$ in $\overline{\Omega}$ for every $\lambda \in (0, \overline{\lambda})$. Moreover, $u_{2,\lambda}$ is unstable for every $\lambda \in (0, \overline{\lambda})$ and there exists $\lambda_n \to 0^+$ such that $u_{2,\lambda_n} \to u_{2,0}$ in $C^{2+\theta}(\overline{\Omega})$ for any $\theta \in (0, \alpha)$ as $n \to \infty$, where $u_{2,0}$ is a positive solution of

$$\begin{cases} -\Delta u = a(x)u^{p-1} & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.9}$$

Remark 1.4. In accordance with Theorems 1.1 and 1.3, a possible positive solutions set of (P_λ) is depicted in Figure 2.

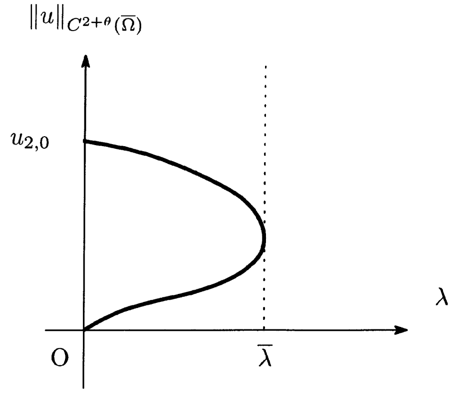

FIGURE 2. A possible bifurcation diagram for (P_λ) when $\int_\Omega a < 0$ and a changes sign.

The outline of this article is the following: in Section 2 we show that nontrivial non-negative solutions of (P_λ) are positive on $\overline{\Omega}$ and that (1.8) is a necessary condition for the existence of positive solutions of (P_λ). In Section 3 we carry out a bifurcation analysis to discuss existence of bifurcating positive solutions to the region $\lambda > 0$ at $(0,0)$. In Section 4 we use variational techniques to discuss multiplicity of positive solutions and their asymptotic profiles as $\lambda \to 0^+$. Finally, in Section 5 we discuss existence of a unbounded subcontinuum of positive solutions of (P_λ) in $\lambda \in \mathbb{R}$. The details of the proofs of Theorems 1.1 and 1.3 appear in [18].

2. Positivity and a necessary condition

We begin this section showing the positivity on $\partial\Omega$ of nontrivial non-negative weak solutions of (P_λ). As mentioned in the Introduction, the boundary point lemma is difficult to apply directly to (P_λ) since $0 < q-1 < 1$. However, by making good use of a comparison principle for a class of nonlinear boundary value problems of concave type, we are able to show that nontrivial non-negative weak solutions of (P_λ) with $\lambda > 0$ are positive on the whole of $\overline{\Omega}$:

Proposition 2.1. *Assume* (1.1). *Then any nontrivial non-negative weak solution of* (P_λ) *is strictly positive on* $\overline{\Omega}$.

Proof. First of all, we note that under (1.1) any nontrivial non-negative weak solution belongs to $X \cap C^\theta(\overline{\Omega})$ for some $\theta \in (0,1)$, cf. Rossi [20, Theorem 2.2]. We consider the following boundary value problem of concave type

$$\begin{cases} -\Delta u = -a_0 u^{p-1} & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = \lambda u^{q-1} & \text{on } \partial\Omega, \end{cases}$$

where $a^- = a^+ - a$, and $a_0 = \sup_\Omega a^-$. A nontrivial non-negative weak solution u_λ of (P_λ) for $\lambda > 0$ satisfies

$$\int_\Omega \nabla u_\lambda \nabla w + a_0 \int_\Omega u_\lambda^{p-1} w - \lambda \int_{\partial\Omega} u_\lambda^{q-1} w \geq 0,$$

for every $w \in X$ such that $w \geq 0$. On the other hand, we consider the following eigenvalue problem:

$$\begin{cases} -\Delta\phi = \sigma\phi & \text{in } \Omega, \\ \frac{\partial\phi}{\partial\mathbf{n}} = \lambda\phi & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

It is easy to see that for any $\lambda > 0$ this problem has a smallest eigenvalue σ_1, which is negative. So, using a positive eigenfunction ϕ_1 associated to σ_1, we deduce that if ε is sufficiently small then $\varepsilon\phi_1$ satisfies

$$\int_\Omega \nabla(\varepsilon\phi_1)\nabla w + a_0 \int_\Omega (\varepsilon\phi_1)^{p-1} w - \lambda \int_{\partial\Omega} (\varepsilon\phi_1)^{q-1} w \leq 0,$$

for every $w \in X$ such that $w \geq 0$. By the comparison principle [16, Proposition A.1], we infer that $\varepsilon\phi_1 \leq u_\lambda$ on $\overline{\Omega}$. In particular, we have $0 < \varepsilon\phi_1 \leq u_\lambda$ on $\partial\Omega$. $\square$

Remark 2.2. Thanks to the positivity property, the assumption $a \in C^\alpha(\overline{\Omega})$, $0 < \alpha < 1$, allows us to prove that under (1.1), any nontrivial non-negative weak solution u of (P_λ) belongs to $C^{2+\theta}(\overline{\Omega})$ for some $\theta \in (0,1)$, by elliptic regularity. Proposition 2.1 will be needed in a combination argument of bifurcation and variational techniques, since our purpose in this paper is to discuss the existence of a classical solution of (P_λ) which is positive in the closure $\overline{\Omega}$.

We prove now that (1.8) is a necessary condition for (P_λ) to have a positive solution for some $\lambda > 0$.

Proposition 2.3. *If (P_λ) has a positive solution for some $\lambda > 0$ then (1.8) is satisfied.*

Proof. Let u be a positive solution of (P_λ). Then we have

$$\int_\Omega \nabla u \nabla w - \int_\Omega a u^{p-1} w - \lambda \int_{\partial\Omega} u^{q-1} w = 0, \quad \forall w \in X.$$

Since $u^{1-p} \in X$, we deduce that

$$\int_\Omega a = \int_\Omega \nabla u \nabla \left(u^{1-p}\right) - \lambda \int_{\partial\Omega} u^{q-1} \frac{1}{u^{p-1}} = (1-p) \int_\Omega u^{-p} |\nabla u|^2 - \lambda \int_{\partial\Omega} u^{-(p-q)} < 0,$$

as desired. $\square$

Remark 2.4. By virtue of Proposition 2.1, under (1.1) we can prove that Proposition 2.3 holds for nontrivial non-negative weak solutions of (P_λ) .

3. A bifurcation analysis

Throughout this section, we assume (1.8). As we shall discuss bifurcation from the zero solution, the following result will be useful (see [17] for a similar proof):

Lemma 3.1. *Assume (1.1). If (λ_n, u_n) are weak solutions of (P_λ) with (λ_n) bounded then $\|u_n\| \to 0$ if and only if $\|u_n\|_{C(\overline{\Omega})} \to 0$.*

We use now a bifurcation technique to show the existence of at least one positive solution of (P_λ) for $\lambda > 0$ close to 0. To this end, we consider positive solutions of the following problem, which corresponds to (P_λ) after the change of variable $w = \lambda^{-\frac{1}{p-q}} u$:

$$\begin{cases} -\Delta w = \lambda^{\frac{p-2}{p-q}} a w^{p-1} & \text{in } \Omega, \\ \frac{\partial w}{\partial n} = \lambda^{\frac{p-2}{p-q}} w^{q-1} & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Proposition 3.2.

(1) If (3.1) *has a sequence of positive solutions* (λ_n, w_n) *such that* $\lambda_n \to 0^+$, $w_n \to c$ *in* $C(\overline{\Omega})$ *and* c *is a positive constant then* $c = c^*$, *where* c^* *is given by* (1.6).

(2) Conversely, (3.1) *has for* $|\lambda|$ *sufficiently small a secondary bifurcation branch* $(\lambda, w(\lambda))$ *of positive solutions (parametrized by* λ*) emanating from the trivial line* $\{(0, c) : c$ *is a positive constant*$\}$ *at* $(0, c^*)$ *and such that, for* $0 < \theta \leq \alpha$, *the mapping* $\lambda \mapsto w(\lambda) \in C^{2+\theta}(\overline{\Omega})$ *is continuous. Moreover, the set* $\{(\lambda, w)\}$ *of positive solutions of* (3.1) *around* $(\lambda, w) = (0, c^*)$ *consists of the union*

$$\{(0, c) : c \textit{ is a positive constant, } |c - c^*| \leq \delta_1\} \cup \{(\lambda, w(\lambda)) : |\lambda| \leq \delta_1\}$$

for some $\delta_1 > 0$.

Proof. The proof is similar to the one of [16, Proposition 5.3]:

(1) Let w_n be positive solutions of (3.1) with $\lambda = \lambda_n$, where $\lambda_n \to 0^+$. By the Green formula we have

$$\int_\Omega a w_n^{p-1} + \int_{\partial\Omega} w_n^{q-1} = 0.$$

Passing to the limit as $n \to \infty$, we deduce the desired conclusion.

(2) We reduce (3.1) to a bifurcation equation in $\mathbb{R}^2$ by the Lyapunov-Schmidt procedure: we use the usual orthogonal decomposition

$$L^2(\Omega) = \mathbb{R} \oplus V,$$

where $V = \{v \in L^2(\Omega) : \int_\Omega v = 0\}$ and the projection $Q : L^2(\Omega) \to V$, given by

$$v = Qu = u - \frac{1}{|\Omega|} \int_\Omega u.$$

The problem of finding a positive solution of (3.1) reduces then to the following problems:

$$\begin{cases} -\Delta v + \frac{\mu}{|\Omega|} \int_{\partial\Omega} (t+v)^{q-1} = \mu Q[a(t+v)^{p-1}] & \text{in } \Omega, \\ \frac{\partial v}{\partial \mathbf{n}} = \mu (t+v)^{q-1} & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

$$\mu \left(\int_\Omega a(t+v)^{p-1} + \int_{\partial\Omega} (t+v)^{q-1} \right) = 0, \tag{3.3}$$

where $\mu = \lambda^{\frac{p-2}{p-q}}$, $t = \frac{1}{|\Omega|} \int_\Omega w$, and $v = w - t$. To solve (3.2) in the framework of Hölder spaces, we set

$$Y = \left\{ v \in C^{2+\theta}(\overline{\Omega}) : \int_\Omega v = 0 \right\},$$

$$Z = \left\{ (\phi, \psi) \in C^\theta(\overline{\Omega}) \times C^{1+\theta}(\partial\Omega) : \int_\Omega \phi + \int_{\partial\Omega} \psi = 0 \right\}.$$

Let $c > 0$ be a constant and $U \subset \mathbb{R} \times \mathbb{R} \times Y$ be a small neighborhood of $(0, c, 0)$. We consider the nonlinear mapping $F : U \to Z$ given by

$$F(\mu, t, v) = \left(-\Delta v - \mu Q[a(t+v)^{p-1}] + \frac{\mu}{|\Omega|} \int_{\partial\Omega} (t+v)^{q-1},\ \frac{\partial v}{\partial \mathbf{n}} - \mu (t+v)^{q-1} \right).$$

The Fréchet derivative F_v of F with respect to v at $(0, c, 0)$ is given by the formula

$$F_v(0, c, 0)v = \left(-\Delta v,\ \frac{\partial v}{\partial \mathbf{n}} \right).$$

Since $F_v(0,c,0)$ is a homeomorphism, the implicit function theorem implies that the set $F(\mu,t,v)=0$ around $(0,c,0)$ consists of a unique C^∞ function $v=v(\mu,t)$ in a neighborhood of $(\mu,t)=(0,c)$ and satisfying $v(0,c)=0$. Now, plugging $v(\mu,t)$ in (3.3), we obtain the bifurcation equation

$$\Phi(\mu,t)=\int_\Omega a(t+v(\mu,t))^{p-1}+\int_{\partial\Omega}(t+v(\mu,t))^{q-1}=0, \quad \text{for } (\mu,t)\simeq(0,c).$$

It is clear that $\Phi(0,c^*)=0$. Differentiating Φ with respect to t at $(0,c^*)$ we get

$$\begin{aligned}\Phi_t(0,c^*)&=\int_\Omega a(p-1)(c^*+v(0,c^*))^{p-2}(1+v_t(0,c^*))\\&\quad+\int_{\partial\Omega}(q-1)(c^*+v(0,c^*))^{q-2}(1+v_t(0,c^*))\\&=(p-1)(c^*)^{p-2}\int_\Omega a(1+v_t(0,c^*))+(q-1)(c^*)^{q-2}\int_{\partial\Omega}(1+v_t(0,c^*)).\end{aligned}$$

Differentiating now (3.2) with respect to t, and plugging $(\mu,t)=(0,c^*)$ therein, we have $v_t(0,c^*)=0$. Hence

$$\Phi_t(0,c^*)=(p-1)(c^*)^{p-2}\left(\int_\Omega a\right)+(q-1)(c^*)^{q-2}|\partial\Omega|=|\partial\Omega|^{\frac{p-2}{p-q}}\left(-\int_\Omega a\right)^{\frac{2-q}{p-q}}(q-p)<0.$$

By the implicit function theorem, the function $w(\lambda)=t(\mu)+v(\mu,t(\mu))$ with $\mu=\lambda^{\frac{p-2}{p-q}}$ satisfies the desired assertion.

□

By considering the transform $u(\lambda)=\lambda^{\frac{1}{p-q}}w(\lambda)$, we get the following result:

Proposition 3.3. *Let $0<\theta\le\alpha$ and $w(\lambda)$ be given by Proposition 3.2. If $\lambda>0$ is sufficiently small then $u(\lambda)=\lambda^{\frac{1}{p-q}}w(\lambda)$ is a positive solution of (P_λ) which satisfies $\lambda^{-\frac{1}{p-q}}u(\lambda)\to c^*$ in $C^{2+\theta}(\overline{\Omega})$ as $\lambda\to 0^+$. In particular, $u(\lambda)\to 0$ in $C^{2+\theta}(\overline{\Omega})$ as $\lambda\to 0^+$.*

4. Variational approach

We associate to (P_λ) the C^1 functional

$$I_\lambda(u):=\frac{1}{2}E(u)-\frac{1}{p}A(u)-\frac{\lambda}{q}B(u), \quad u\in X,$$

where

$$E(u)=\int_\Omega|\nabla u|^2, \quad A(u)=\int_\Omega a(x)|u|^p, \quad \text{and} \quad B(u)=\int_{\partial\Omega}|u|^q.$$

Let us recall that $X=H^1(\Omega)$ is equipped with the usual norm $\|u\|=\left[\int_\Omega\left(|\nabla u|^2+u^2\right)\right]^{\frac{1}{2}}$. We denote by $\rightharpoonup$ the weak convergence in X.

The following result will be used repeatedly in this section.

Lemma 4.1.

(1) If (u_n) is a sequence such that $u_n\rightharpoonup u_0$ in X and $\limsup_n E(u_n)\le 0$ then u_0 is a constant and $u_n\longrightarrow u_0$ in X.
(2) Assume (1.8). If $v\ne 0$ and $A(v)\ge 0$, then v is not a constant.

Proof.

(1) Since $u_n \rightharpoonup u_0$ in X and E is weakly lower semicontinuous, we have $E(u_0) \leq \liminf_n E(u_n)$, so that
$$0 \leq E(u_0) \leq \liminf_n E(u_n) \leq \limsup_n E(u_n) \leq 0.$$
Hence, $E(u_0) = 0$, which implies that u_0 is a constant. Assume $u_n \not\to u_0$ in X. Then $E(u_0) < \limsup_n E(u_n) \leq 0$, which is a contradiction. Therefore $u_n \to u_0$ in X.

(2) If $v_0 \neq 0$ is a constant then $0 \leq A(v_0) = |v_0|^p \int_\Omega a < 0$, a contradiction.

□

Now, in addition to (1.1) and (1.8), we assume that a changes sign. Moreover, we assume $p < \frac{2N}{N-2}$ if $N > 2$. We shall prove the existence of two positive solutions of (P_λ) for $0 < \lambda < \overline{\lambda}$ and characterize their asymptotic profiles as $\lambda \to 0^+$. To this end, we use the Nehari manifold and the fibering maps associated to I_λ. Let us introduce some useful subsets of X:
$$E^+ = \{u \in X : E(u) > 0\},$$
$$A^\pm = \{u \in X : A(u) \gtrless 0\}, \quad A_0 = \{u \in X : A(u) = 0\}, \quad A_0^\pm = A^\pm \cup A_0,$$
$$B^+ = \{u \in X : B(u) > 0\}.$$
The Nehari manifold associated to I_λ is given by
$$N_\lambda := \{u \in X \setminus \{0\} : \langle I_\lambda'(u), u\rangle = 0\} = \{u \in X \setminus \{0\} : E(u) = A(u) + \lambda B(u)\}.$$
We shall use the splitting
$$N_\lambda = N_\lambda^+ \cup N_\lambda^- \cup N_\lambda^0,$$
where
$$N_\lambda^\pm := \{u \in N_\lambda : \langle J_\lambda'(u), u\rangle \gtrless 0\} = \left\{u \in N_\lambda : E(u) \lessgtr \lambda \frac{p-q}{p-2} B(u)\right\}$$
$$= \left\{u \in N_\lambda : E(u) \gtrless \frac{p-q}{2-q} A(u)\right\},$$
and
$$N_\lambda^0 = \{u \in N_\lambda : \langle J_\lambda'(u), u\rangle = 0\}.$$
Note that any nontrivial weak solution of (P_λ) belongs to N_λ. Furthermore, it follows from the implicit function theorem that $N_\lambda \setminus N_\lambda^0$ is a C^1 manifold and every critical point of the restriction of I_λ to this manifold is a critical point of I_λ (see for instance [6, Theorem 2.3]).

To analyse the structure of $N_\lambda^\pm$, we consider the fibering maps corresponding to I_λ for $u \neq 0$ in the following way:
$$j_u(t) := I_\lambda(tu) = \frac{t^2}{2}E(u) - \frac{t^p}{p}A(u) - \lambda\frac{t^q}{q}B(u), \quad t > 0.$$
It is easy to see that
$$j_u'(1) = 0 \lessgtr j_u''(1) \iff u \in N_\lambda^\pm,$$
and more generally,
$$j_u'(t) = 0 \lessgtr j_u''(t) \iff tu \in N_\lambda^\pm.$$
Having this characterisation in mind, we look for conditions under which j_u has a critical point. Set
$$i_u(t) := t^{-q}j_u(t) = \frac{t^{2-q}}{2}E(u) - \frac{t^{p-q}}{p}A(u) - \lambda B(u), \quad t > 0.$$

Let $u \in E^+ \cap A^+ \cap B^+$. Then i_u has a global maximum $i_u(t^*)$ at some $t^* > 0$, and moreover, t^* is unique. If $i_u(t^*) > 0$, then j_u has a global maximum which is positive and a local minimum which is negative. Moreover, these are the only critical points of j_u. We

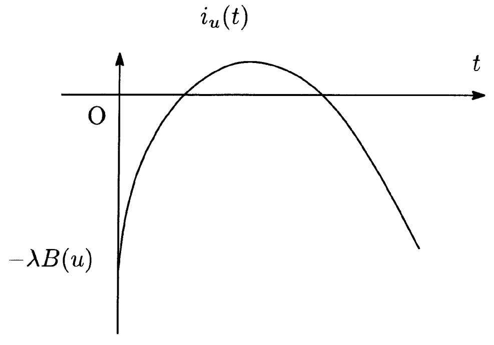

FIGURE 3. The case $i_u(t^*) > 0$.

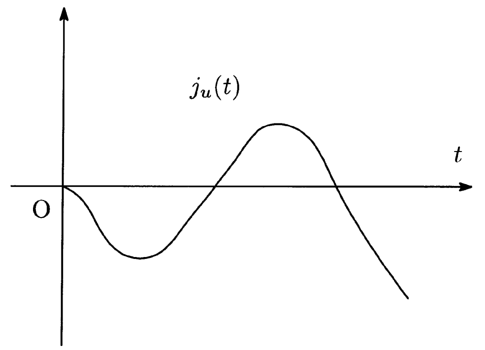

FIGURE 4. A case of j_u having a global maximum and a local minimum.

shall require a condition on λ that provides $i_u(t^*) > 0$. Note that

$$i'_u(t) = \frac{2-q}{2} t^{1-q} E(u) - \frac{p-q}{p} t^{p-q-1} A(u) = 0$$

if and only if

$$t = t^* := \left(\frac{p(2-q)E(u)}{2(p-q)A(u)} \right)^{\frac{1}{p-2}}.$$

Moreover

$$i_u(t^*) = \frac{p-2}{2(p-q)} \left(\frac{p(2-q)}{2(p-q)} \right)^{\frac{2-q}{p-2}} \frac{E(u)^{\frac{p-q}{p-2}}}{A(u)^{\frac{2-q}{p-2}}} - \frac{\lambda}{q} B(u) > 0$$

if and only if

$$0 < \lambda^{\frac{p-2}{p-q}} < C_{pq} \frac{E(u)}{B(u)^{\frac{p-2}{p-q}} A(u)^{\frac{2-q}{p-q}}}, \tag{4.1}$$

where $C_{pq} = \left(\frac{q(p-2)}{2(p-q)}\right)^{\frac{p-2}{p-q}} \left(\frac{p(2-q)}{2(p-q)}\right)^{\frac{2-q}{p-q}}$. Note that $F(u) = \frac{E(u)}{B(u)^{\frac{p-2}{p-q}} A(u)^{\frac{2-q}{p-q}}}$ satisfies $F(tu) = F(u)$ for $t > 0$, i.e. F is homogeneous of order 0.

We deduce then the following result, which provides sufficient conditions for the existence of critical points of j_u:

Proposition 4.2. *The following assertions hold:*

(1) If either $u \in E^+ \cap A_0^- \cap B^+$ or $u \in A^- \cap B^+$ then $j_u(t)$ has a negative global minimum at some $t_1 > 0$, i.e. $j_u'(t_1) = 0 < j_u''(t_1)$, and $j_u(t) > j_u(t_1)$ for $t \neq t_1$. Moreover, t_1 is the unique critical point of j_u and $j_u(t) \to \infty$ as $t \to \infty$.

(2) If $u \in E^+ \cap A^+ \cap B_0$ then $j_u(t)$ has a positive global maximum at some $t_2 > 0$, i.e. $j_u'(t_2) = 0 > j_u''(t_2)$ and $j_u(t) < j_u(t_2)$ for $t \neq t_1$. Moreover, t_2 is the unique critical point of j_u and $j_u(t) \to -\infty$ as $t \to \infty$.

(3) Assume (1.8). *If we set*

$$\lambda_0^{\frac{p-2}{p-q}} = \inf\{E(u) : u \in E^+ \cap A^+ \cap B^+,\ C_{pq}^{-1} B(u)^{\frac{p-2}{p-q}} A(u)^{\frac{2-q}{p-q}} = 1\}, \tag{4.2}$$

then $\lambda_0 > 0$. Moreover, for any $0 < \lambda < \lambda_0$ and $u \in E^+ \cap A^+ \cap B^+$ the map j_u has a negative local minimum at $t_1 > 0$ and a positive global maximum at $t_2 > t_1$. Furthermore, t_1, t_2 are the only critical points of j_u and $j_u(t) \to -\infty$ as $t \to \infty$ (see Figure 4).

Proof. Assertions (1) and (2) are straightforward from the definition of j_u. We prove now assertion (3). First, we show that $\lambda_0 > 0$. Assume $\lambda_0 = 0$, so that we can choose $u_n \in E^+ \cap A^+ \cap B^+$ satisfying

$$E(u_n) \longrightarrow 0, \quad \text{and} \quad C_{pq}^{-1} B(u_n)^{\frac{p-2}{p-q}} A(u_n)^{\frac{2-q}{p-q}} = 1.$$

If (u_n) is bounded in X then we may assume that $u_n \rightharpoonup u_0$ for some $u_0 \in X$ and $u_n \to u_0$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$. It follows from Lemma 4.1(1) that u_0 is a constant and $u_n \to u_0$ in X. From $u_n \in A^+$ we deduce that $u_0 \in A_0^+$. In addition, we have

$$C_{pq}^{-1} B(u_0)^{\frac{p-2}{p-q}} A(u_0)^{\frac{2-q}{p-q}} = 1,$$

so that $u_0 \not\equiv 0$. From Lemma 4.1(2) we get a contradiction.

Let us assume now that $\|u_n\| \to \infty$. Set $v_n = \frac{u_n}{\|u_n\|}$, so that $\|v_n\| = 1$. We may assume that $v_n \rightharpoonup v_0$ and $v_n \to v_0$ in $L^p(\Omega)$. Since $E(v_n) \to 0$ and $v_n \in A^+$, we have $v_n \to v_0$ in X, v_0 is a constant, and $v_0 \in A_0^+$. In particular, $\|v_0\| = 1$, i.e. $v_0 \not\equiv 0$. Lemma 4.1 provides again a contradiction.

Finally, for any $u \in E^+ \cap A^+ \cap B^+$ we have

$$\lambda_0^{\frac{p-2}{p-q}} \leq C_{pq} \frac{E(u)}{B(u)^{\frac{p-2}{p-q}} A(u)^{\frac{2-q}{p-q}}}.$$

Thus, if $0 < \lambda < \lambda_0$ then $i_u(t^*) > 0$ from (4.1). This completes the proof of assertion (3). $\square$

Proposition 4.3. *We have, for $0 < \lambda < \lambda_0$:*

(1) N_λ^0 is empty.
(2) $N_\lambda^\pm$ are non-empty.

Proof.

(1) From Proposition 4.2 it follows that there is no $t > 0$ such that $j_u'(t) = j_u''(t) = 0$, i.e. N_λ^0 is empty.

(2) Consider the following eigenvalue problem

$$\begin{cases} -\Delta\varphi = \lambda a(x)\varphi & \text{in } \Omega, \\ \frac{\partial\varphi}{\partial\mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases}$$

Under (1.8) it is known that this problem has a unique positive principal eigenvalue λ_N with a positive principal eigenfunction φ_N. From $\varphi_N > 0$ on $\partial\Omega$ and the fact that φ_N is not a constant, we deduce that $\varphi_N \in E^+ \cap A^+ \cap B^+$. Since $0 < \lambda < \lambda_0$, Proposition 4.2(3) provides the desired conclusion.

□

The following result provides some properties of N_λ^+:

Lemma 4.4. *Let $0 < \lambda < \lambda_0$. Then, we have the following two assertions:*

(1) N_λ^+ is bounded in X.
(2) $I_\lambda(u) < 0$ for any $u \in N_\lambda^+$ and moreover $t > 1$ if $j_u'(t) > 0$.

Proof.

(1) Assume $(u_n) \subset N_\lambda^+$ and $\|u_n\| \to \infty$. Set $v_n = \frac{u_n}{\|u_n\|}$. It follows that $\|v_n\| = 1$, so we may assume that $v_n \rightharpoonup v_0$, $B(v_n)$ is bounded, and $v_n \to v_0$ in $L^p(\Omega)$ (implying $A(v) \to A(v_0)$). Since $u_n \in N_\lambda^+$, we see that

$$E(v_n) < \lambda \frac{p-q}{p-2} B(v_n)\|u_n\|^{q-2},$$

and thus $\limsup_n E(v_n) \le 0$. Lemma 4.1(1) yields that v_0 is a constant and $v_n \to v_0$ in X. Consequently, $\|v_0\| = 1$, and v_0 is a non-zero constant. However, since $u_n \in N_\lambda$, we see that

$$0 \le E(u_n) = A(u_n) + \lambda B(u_n),$$

and it follows that

$$0 \le A(v_n) + \lambda B(v_n)\|u_n\|^{q-p}.$$

Passing to the limit as $n \to \infty$, we deduce $0 \le A(v_0)$. Lemma 4.1(2) leads us to a contradiction. Therefore N_λ^+ is bounded in X.

(2) Let $u \in N_\lambda^+$. Then

$$0 \le E(u) < \lambda \frac{p-q}{p-2} B(u),$$

so that $B(u) > 0$. First we assume that u is not a constant. In this case $E(u) > 0$. If $A(u) > 0$ then Proposition 4.2(3) tells us that $I_\lambda(u) < 0$ and $t > 1$ if $j_u'(t) > 0$. On the other hand, if $A(u) \le 0$ then $u \in E^+ \cap A_0^- \cap B^+$. So Proposition 4.2(1) gives the same conclusion. Assume now that u is a constant. In this case $A(u) = |u|^p \int_\Omega a < 0$, so that $u \in A^- \cap B^+$. Proposition 4.2(1) again yields the desired conclusion.

□

Next we prove that $\inf_{N_\lambda^+} I_\lambda$ is achieved by some $u_{1,\lambda} > 0$ for $\lambda \in (0, \lambda_0)$, which implies the estimate $\overline{\lambda} \ge \lambda_0$. Furthermore, we can show that $u_{1,\lambda}$ is in fact the minimal positive solution of (P_λ) for $\lambda > 0$ sufficiently small.

Proposition 4.5. *For any $0 < \lambda < \lambda_0$, there exists $u_{1,\lambda}$ such that $I_\lambda(u_{1,\lambda}) = \min_{N_\lambda^+} I_\lambda$. In particular, $u_{1,\lambda}$ is a positive solution of (P_λ).*

Proof. Let $0 < \lambda < \lambda_0$. We consider a minimizing sequence $(u_n) \subset N_\lambda^+$, i.e.

$$I_\lambda(u_n) \longrightarrow \inf_{N_\lambda^+} I_\lambda < 0.$$

Since (u_n) is bounded in X, we may assume that $u_n \rightharpoonup u_0$, $u_n \to u_0$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$. It follows that

$$I_\lambda(u_0) \leq \liminf_n I_\lambda(u_n) = \inf_{N_\lambda^+} I_\lambda(u) < 0,$$

so that $u_0 \not\equiv 0$. We claim that $u_n \to u_0$ in X. We have two possibilities:

- If u_0 is a constant, then $0 = E(u_0) \leq \lambda \frac{p-q}{p-2} B(u_0)$. If $B(u_0) = 0$ then $u_0 = 0$ on $\partial\Omega$, so that $u_0 = 0$ in Ω, which yields a contradiction. Hence $B(u_0) > 0$. In this case, we have $A(u_0) = |u_0|^p \int_\Omega a < 0$, so that $u_0 \in A^- \cap B^+$. Proposition 4.2(1) implies that $t_1 u_0 \in N_\lambda^+$ and j_{u_0} has a global minimum at t_1. If $u_n \not\to u_0$ then

$$I_\lambda(t_1 u_0) = j_{u_0}(t_1) \leq j_{u_0}(1) < \liminf_n j_{u_n}(1) = \liminf_n I_\lambda(u_n) = \inf_{N_\lambda^+} I_\lambda, \tag{4.3}$$

  which is a contradiction since $t_1 u_0 \in N_\lambda^+$. Therefore $u_n \to u_0$.

- If u_0 is not a constant then $E(u_0) > 0$ and $B(u_0) > 0$. So either $u_0 \in E^+ \cap A_0^- \cap B^+$ or $u_0 \in E^+ \cap A^+ \cap B^+$. In the first case, j_{u_0} has a global minimum point t_1 and we can argue as in the previous case. In the second case, since $0 < \lambda < \lambda_0$, Proposition 4.2 yields that $t_1 u_0 \in N_\lambda^+$ for some $t_1 > 0$. Assume $u_n \not\to u_0$. If $1 < t_1$ then we have again

$$I_\lambda(t_1 u_0) = j_{u_0}(t_1) \leq j_{u_0}(1) < \liminf_n j_{u_n}(1) = \liminf_n I_\lambda(u_n) = \inf_{N_\lambda^+} I_\lambda, \tag{4.4}$$

  If $t_1 < 1$ then $j'_{u_n}(t_1) < 0$ for every n, so that $j'_{u_0}(t_1) < \liminf j'_{u_n}(t_1) \leq 0$, which is a contradiction. Therefore $u_n \to u_0$.

Now, since $u_n \to u_0$ we have $j'_{u_0}(1) = 0 \leq j''_{u_0}(1)$. But $j''_{u_0}(1) = 0$ is impossible by Proposition 4.3(1). Thus $u_0 \in N_\lambda^+$ and $I_\lambda(u_0) = \inf_{N_\lambda^+} I_\lambda$. □

Remark 4.6. From Proposition 4.5 we derive $\overline{\lambda} \geq \lambda_0$.

Next we obtain a second nontrivial non-negative weak solution of (P_λ), which achieves $\inf_{N_\lambda^-} I_\lambda$ for $\lambda \in (0, \lambda_0)$. The following result provides some properties of N_λ^-:

Lemma 4.7. *Let $0 < \lambda < \lambda_0$. Then we have $I_\lambda(u) > 0$ for any $u \in N_\lambda^-$. Moreover $t < 1$ if $j'_u(t) > 0$.*

Proof. If $u \in N_\lambda^-$ then $A(u) > 0$ and u is not a constant from Lemma 4.1(2). It follows immediately that $E(u) > 0$. If $B(u) > 0$, then, by Proposition 4.2(3), we have that $I_\lambda(u) > 0$ and $t < 1$ if $j'_u(t) > 0$. If $B(u) = 0$, then Proposition 4.2(2) provides the same conclusion. □

Proposition 4.8. *For any $\lambda \in (0, \lambda_0)$, there exists $u_{2,\lambda}$ such that $I_\lambda(u_{2,\lambda}) = \min_{N_\lambda^-} I_\lambda$. In particular, $u_{2,\lambda}$ is a positive solution of (P_λ).*

Proof. Since $I_\lambda(u) > 0$ for $u \in N_\lambda^-$, we can choose $u_n \in N_\lambda^-$ such that

$$I_\lambda(u_n) \longrightarrow \inf_{N_\lambda^-} I_\lambda(u) \geq 0.$$

We claim that (u_n) is bounded in X. Indeed, there exists $C > 0$ such that $I_\lambda(u_n) \leq C$. Since $u_n \in N_\lambda$, we deduce

$$\left(\frac{1}{2} - \frac{1}{p}\right) E(u_n) - \lambda \left(\frac{1}{q} - \frac{1}{p}\right) B(u_n) = I_\lambda(u_n) \leq C.$$

Assume $\|u_n\| \to \infty$ and set $v_n = \frac{u_n}{\|u_n\|}$, so that $\|v_n\| = 1$. We may assume that $v_n \rightharpoonup v_0$, and $v_n \to v_0$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$. Then, from

$$\left(\frac{1}{2} - \frac{1}{p}\right) E(v_n) \leq \lambda \left(\frac{1}{q} - \frac{1}{p}\right) B(v_n)\|u_n\|^{q-2} + \frac{C}{\|u_n\|^2},$$

we infer that $\limsup_n E(v_n) \leq 0$. Lemma 4.1(1) yields that v_0 is a constant, and $v_n \to v_0$ in X, which implies $\|v_0\| = 1$. However, since $u_n \in N_\lambda^-$, we observe that

$$E(v_n)\|u_n\|^{2-p} < \frac{p-q}{2-q} A(v_n).$$

Passing to the limit $n \to \infty$, we get $0 \leq A(v_0)$, which is contradictory by Lemma 4.1(2). Hence (u_n) is bounded. We may then assume that $u_n \rightharpoonup u_0$, and $u_n \to u_0$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$. We claim that $u_n \to u_0$ in X. Assume $u_n \not\to u_0$. Then, since $u_n \in N_\lambda^-$, we deduce

$$0 \leq E(u_0) < \liminf_n E(u_n) \leq \liminf_n \frac{p-q}{2-q} A(u_n) = \frac{p-q}{2-q} A(u_0).$$

This implies that u_0 is not a constant by Lemma 4.1(2), so that $E(u_0) > 0$. Since $u_0 \in E^+ \cap A^+$, Proposition 4.2 tells us that there exists $t_2 > 0$ such that $t_2 u_0 \in N_\lambda^-$. Moreover, $0 = j'_{u_0}(t_2) < \liminf_n j'_{u_n}(t_2)$, since $u_n \not\to u_0$. We deduce that $j'_{u_n}(t_2) > 0$ for n large enough. Since $u_n \in N_\lambda^-$, we have $t_2 < 1$ from Lemma 4.7. Then, we observe that

$$I_\lambda(t_2 u_0) = j_{u_0}(t_2) < \liminf_n j_{u_n}(t_2) \leq \liminf_n j_{u_n}(1) = \liminf_n I_\lambda(u_n) = \inf_{N_\lambda^-} I_\lambda.$$

This is a contradiction, which implies that $u_n \to u_0$ and $I_\lambda(u_n) \to I_\lambda(u_0) = \gamma$.

Now we verify that $u_0 \neq 0$. Assume $u_0 = 0$. Then, since $u_n \in N_\lambda$, we have

$$E(v_n)\|u_n\|^{2-q} = A(v_n)\|u_n\|^{p-q} + \lambda B(v_n),$$

where $v_n = \frac{u_n}{\|u_n\|}$. We may assume again that $v_n \rightharpoonup v_0$ and $v_n \to v_0$ in $L^q(\partial\Omega)$ and $L^p(\Omega)$. Passing to the limit as $n \to \infty$, we obtain $0 = \lambda B(v_0)$, so that $v_0 = 0$ on $\partial\Omega$. On the other hand, we observe that

$$0 < I_\lambda(u_n) = \frac{1}{2}E(u_n) - \frac{1}{p}A(u_n) - \frac{\lambda}{q}B(u_n).$$

Since $u_n \in N_\lambda$, we deduce

$$\left(\frac{1}{q} - \frac{1}{2}\right) E(v_n) \leq \left(\frac{1}{q} - \frac{1}{p}\right) A(v_n)\|u_n\|^{p-2}.$$

From the assumption $u_n \to 0$ in X, it follows that $\limsup E(v_n) \leq 0$. By Lemma 4.1(1) we get that v_0 is a constant, and $v_n \to v_0$ in X, so that $\|v_0\| = 1$. Since v_0 is a constant and $v_0 = 0$ on $\partial\Omega$, we have $v_0 = 0$ in Ω. This is a contradiction, as desired.

Finally, since $u_n \to u_0$ in X we have $j'_{u_0}(1) = 0 \geq j''_{u_0}(1)$. But $j''_{u_0}(1) = 0$ is impossible by Proposition 4.3(1). Thus $u_0 \in N_\lambda^-$ and $I_\lambda(u_0) = \inf_{N_\lambda^-} I_\lambda$.

□

We discuss now the asymptotic profiles of $u_{1,\lambda}, u_{2,\lambda}$ as $\lambda \to 0^+$. The following lemma is concerned with the behavior of positive solutions in N_λ^+ as $\lambda \to 0^+$:

Proposition 4.9. *If u_λ is a positive solution of (P_λ) such that $u_\lambda \in N_\lambda^+$ for $\lambda > 0$ sufficiently small then $u_\lambda \to 0$ in X as $\lambda \to 0^+$. Moreover there holds $\lambda^{-\frac{1}{p-q}} u_\lambda \to c^*$ in $C^{2+\theta}(\overline{\Omega})$ for any $\theta \in (0, \alpha)$ as $\lambda \to 0^+$.*

Proof. First we show that u_λ remains bounded in X as $\lambda \to 0^+$. Indeed, assume that $\|u_\lambda\| \to \infty$ and set $v_\lambda = \frac{u_\lambda}{\|u_\lambda\|}$. We may then assume that for some $v_0 \in X$ we have $v_\lambda \rightharpoonup v_0$ in X, and $v_\lambda \to v_0$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$. Since $u_\lambda \in N_\lambda$, we have

$$E(v_\lambda)\|u_\lambda\|^{2-p} = A(v_\lambda) + \lambda B(v_\lambda)\|u_\lambda\|^{q-p}.$$

Passing to the limit as $\lambda \to 0^+$, we obtain $A(v_0) = 0$. From $u_\lambda \in N_\lambda^+$ we have

$$E(v_\lambda) < \lambda \frac{p-q}{p-2} B(v_\lambda)\|u_\lambda\|^{q-2},$$

so that $\limsup_\lambda E(v_\lambda) \le 0$. By Lemma 4.1(1) we infer that v_0 is a constant and $v_\lambda \to v_0$ in X, so that $\|v_0\| = 1$, i.e. $v_0 \neq 0$. This is contradictory with Lemma 4.1(2), and therefore u_λ stays bounded in X as $\lambda \to 0^+$.

Hence we may assume that $u_\lambda \rightharpoonup u_0$ in X and $u_\lambda \to u_0$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$ as $\lambda \to 0^+$. Since $u_\lambda \in N_\lambda^+$, we observe that

$$E(u_\lambda) < \lambda \frac{p-q}{p-2} B(u_\lambda).$$

Passing to the limit as $\lambda \to 0^+$, we get $\limsup_\lambda E(u_\lambda) \le 0$. Lemma 4.1(2) provides that u_0 is a constant and $u_\lambda \to u_0$ in X. Since $u_\lambda \in N_\lambda$, we have

$$E(u_\lambda) = A(u_\lambda) + \lambda B(u_\lambda).$$

which implies $A(u_0) = 0$, so that $u_0 = 0$ from Lemma 4.1(2). Therefore $u_\lambda \to 0$ in X as $\lambda \to 0^+$.

Now we obtain the asymptotic profile of u_λ as $\lambda \to 0^+$. Let $w_\lambda = \lambda^{-\frac{1}{p-q}} u_\lambda$. We claim that w_λ remains bounded in X as $\lambda \to 0^+$. Indeed, since $u_\lambda \in N_\lambda^+$, we have

$$E(w_\lambda) < \frac{p-q}{p-2} \lambda^{\frac{p-2}{p-q}} B(w_\lambda).$$

Let us assume that $\|w_\lambda\| \to \infty$ and set $\psi_\lambda = \frac{w_\lambda}{\|w_\lambda\|}$. We may assume that $\psi_\lambda \rightharpoonup \psi_0$ and $\psi_\lambda \to \psi_0$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$. It follows that

$$E(\psi_\lambda) < \frac{p-q}{p-2} \lambda^{\frac{p-2}{p-q}} B(\psi_\lambda)\|w_\lambda\|^{q-2},$$

so that $\limsup_\lambda E(\psi_\lambda) \le 0$. By Lemma 4.1(1) we infer that ψ_0 is a constant and $\psi_\lambda \to \psi_0$ in X. On the other hand, from $u_\lambda \in N_\lambda$ it follows that

$$0 \le A(u_\lambda) + \lambda B(u_\lambda),$$

so that

$$-B(\psi_\lambda)\|w_\lambda\|^{q-p} \le A(\psi_\lambda).$$

Taking the limit as $\lambda \to 0^+$ we get $0 \le A(\psi_0)$, which contradicts Lemma 4.1(2). Hence w_λ stays bounded in X as $\lambda \to 0^+$ and we may assume that $w_\lambda \rightharpoonup w_0$ in X and $w_\lambda \to w_0$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$. It follows that $\limsup_\lambda E(w_\lambda) \le 0$, and by Lemma 4.1(1) we get that w_0 is a constant and $w_\lambda \to w_0$ in X.

It remains to show that $w_0 = c^*$. We note that w_λ satisfies

$$\int_\Omega \nabla w_\lambda \nabla w - \lambda^{\frac{p-2}{p-q}} \int_\Omega a w_\lambda^{p-1} w - \lambda^{\frac{p-2}{p-q}} \int_{\partial\Omega} w_\lambda^{q-1} w = 0, \quad \forall w \in X, \tag{4.5}$$

since u_λ is a weak solution of (P_λ). Taking $w=1$, we see that

$$\int_\Omega a w_\lambda^{p-1} + \int_{\partial\Omega} w_\lambda^{q-1} = 0.$$

Passing to the limit as $\lambda \to 0^+$, we see that either $w_0 = 0$ or $w_0 = c^*$. However, taking $w = \frac{1}{w_\lambda^{q-1}}$ in (4.5) we obtain

$$0 > -(q-1)\int_\Omega w_\lambda^{-q}|\nabla w_\lambda|^2 = \lambda^{\frac{p-2}{p-q}}\left(\int_\Omega a w_\lambda^{p-q} + |\partial\Omega|\right),$$

so that

$$|\partial\Omega| < -\int_\Omega a w_\lambda^{p-q}.$$

It is clear then that $w_0 \neq 0$, i.e. $w_0 = c^*$, and consequently we obtain $\lambda^{-\frac{1}{p-q}}u_\lambda \to c^*$ in X. By a standard bootstrap argument, we get the desired conclusion. □

We turn now to the asymptotic behavior of $u_{2,\lambda}$ as $\lambda \to 0^+$. We shall prove initially that solutions in N_λ^- are bounded away from zero as $\lambda \to 0^+$:

Lemma 4.10. *If u_λ is a positive solution of (P_λ) such that $u_\lambda \in N_\lambda^-$ for $\lambda > 0$ sufficiently small then $\|u_\lambda\| \geq C$ for some constant $C > 0$ as $\lambda \to 0^+$.*

Proof. Assume by contradiction that (u_n) is a sequence of positive solutions of (P_{λ_n}) with $\lambda_n \to 0^+$, $u_n \in N_{\lambda_n}^-$ and $\|u_n\| \to 0$. Then, since $u_n \in N_{\lambda_n}^-$, we deduce

$$E(v_n) < \frac{p-q}{2-q}A(v_n)\|u_n\|^{p-2},$$

where $v_n = \frac{u_n}{\|u_n\|}$. We may assume that $v_n \rightharpoonup v_0$ in X and $v_n \to v_0$ in $L^p(\Omega)$. It follows that $\limsup E(v_n) \leq 0$. By Lemma 4.1(1) we get that v_0 is a constant and $v_n \to v_0$ in X, so that $\|v_0\| = 1$. On the other hand, we see that $A(v_n) > 0$, since $u_n \in N_{\lambda_n}^-$. We obtain then $0 \leq A(v_0)$, which is a contradiction with Lemma 4.1(2). □

We prove now that $u_{2,\lambda}$ is bounded in X as $\lambda \to 0^+$:

Lemma 4.11. *There exists a constant $C > 0$ such that $C^{-1} \leq \|u_{2,\lambda}\| \leq C$ as $\lambda \to 0^+$.*

Proof. By Lemma 4.10 we know that $\|u_{2,\lambda}\| \geq C^{-1}$ for some $C > 0$ as $\lambda \to 0^+$. We show now that $u_{2,\lambda}$ is bounded in X as $\lambda \to 0^+$. First, we show that there exists a constant $C_1 > 0$ such that $I_\lambda(u_{2,\lambda}) \leq C_1$ for every $\lambda \in (0,\lambda_0)$. To this end, we consider the following eigenvalue problem with the Dirichlet boundary condition.

$$\begin{cases} -\Delta\varphi = \lambda a(x)\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.6}$$

We denote by φ_D a positive eigenfunction associated with the positive principal eigenvalue λ_D. Multiplying (4.6) by φ_D^{p-1} we see that $\varphi_D \in A^+$. Thus $\varphi_D \in E^+ \cap A^+ \cap B_0$ and

$$j_{\varphi_D}(t) = \frac{t^2}{2}E(\varphi_D) - \frac{t^p}{p}A(\varphi_D),$$

so that j_{φ_D} has a global maximum at some $t_2 > 0$, which implies $t_2\varphi_D \in N_\lambda^-$. Moreover, neither j_{φ_D} nor $t_2\varphi_D$ depend on $\lambda \in (0,\lambda_0)$. Let $C_1 = j_{\varphi_D}(t_2) = I_\lambda(t_2\varphi_D) > 0$. Since $t_2\varphi_D \in N_\lambda^-$, we deduce that $I_\lambda(u_{2,\lambda}) \leq C_1$.

Assume now that $\|u_{2,\lambda}\| \to \infty$ as $\lambda \to 0^+$ and set $v_\lambda = \frac{u_{2,\lambda}}{\|u_{2,\lambda}\|}$. We may assume that $v_\lambda \rightharpoonup v_0$ and $v_\lambda \to v_0$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$. Since

$$0 \le E(u_{2,\lambda}) < \frac{p-q}{2-q} A(u_{2,\lambda}),$$

it follows that $A(v_\lambda) > 0$. Passing to the limit as $\lambda \to 0^+$, we get $A(v_0) \ge 0$. However, we will see that the condition $I_\lambda(u_{2,\lambda}) \le C_1$ leads us to a contradiction. Indeed, since $u_{2,\lambda} \in N_\lambda$, we deduce

$$\left(\frac{1}{2} - \frac{1}{p}\right) E(u_{2,\lambda}) - \left(\frac{1}{q} - \frac{1}{p}\right) \lambda B(u_{2,\lambda}) = I_\lambda(u_{2,\lambda}) \le C_1.$$

Hence

$$\left(\frac{1}{2} - \frac{1}{p}\right) E(v_\lambda) \le \left(\frac{1}{q} - \frac{1}{p}\right) \lambda B(v_\lambda) \|u_{2,\lambda}\|^{q-2} + C_1 \|u_{2,\lambda}\|^{-2}.$$

Letting $\lambda \to 0^+$ we obtain $\limsup_\lambda E(v_\lambda) \le 0$, and by Lemma 4.1 we infer that v_0 is a constant and $v_\lambda \to v_0$ in X. In particular, $\|v_0\| = 1$, which contradicts Lemma 4.1(2). The proof is now complete. $\square$

We establish now (up to a subsequence) the precise limiting behavior of $u_{2,\lambda}$:

Proposition 4.12. *There exists a sequence $\lambda_n \to 0^+$ such that $u_{2,\lambda_n} \to u_{2,0}$ in $C^{2+\theta}(\overline{\Omega})$ for any $\theta \in (0, \alpha)$, where $u_{2,0}$ is a positive solution of* (1.9).

Proof. Since $u_{2,\lambda}$ stays bounded in X as $\lambda \to 0^+$, up to a subsequence, we have $u_{2,\lambda} \rightharpoonup u_{2,0}$, and $u_{2,\lambda} \to u_{2,0}$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$ as $\lambda \to 0^+$. Since $u_{2,\lambda}$ is a weak solution of (P_λ), we have

$$\int_\Omega \nabla u_{2,\lambda} \nabla w - \int_\Omega a u_{2,\lambda}^{p-1} w - \lambda \int_{\partial\Omega} u_{2,\lambda}^{q-1} w = 0, \quad \forall w \in X.$$

Letting $\lambda \to 0^+$, we get

$$\int_\Omega \nabla u_{2,0} \nabla w - \int_\Omega a u_{2,0}^{p-1} w = 0, \quad \forall w \in X,$$

i.e. $u_{2,0}$ is a non-negative weak solution of (1.9). If $u_{2,0} \equiv 0$ then, from

$$E(u_{2,\lambda}) < \frac{p-q}{2-q} A(u_{2,\lambda}) \quad \text{and} \quad A(u_{2,0}) = 0,$$

we deduce that $\limsup_\lambda E(u_{2,\lambda}) \le 0$. By Lemma 4.1(1) we infer that u_0 is a constant and $u_{2,\lambda} \to u_{2,0} = 0$ in X, which contradicts Lemma 4.11.

Finally, since $u_{2,0} \in C^{2+\alpha}(\overline{\Omega})$, and $u_{2,0} > 0$ in $\overline{\Omega}$ by the weak maximum principle and the boundary point lemma, we infer that $u_{2,0}$ is a positive solution of (1.9). By a standard bootstrap argument, we obtain the desired conclusion. $\square$

We shall consider now the Palais-Smale condition for I_λ. Let us recall that I_λ satisfies the Palais-Smale condition if any sequence such that $(I_\lambda(u_n))$ is bounded and $I'_\lambda(u_n) \to 0$ in X' has a convergent subsequence.

Proposition 4.13. *I_λ satisfies the Palais-Smale condition for any $\lambda > 0$..*

Proof. Let (u_n) be a Palais-Smale sequence for I_λ. Then

$$(I_\lambda(u_n)) \text{ is bounded} \quad \text{and} \quad I'_\lambda(u_n)\phi = o(1)\|\phi\| \quad \forall \phi \in X.$$

In particular, we have

$$\left(\frac{1}{2} - \frac{1}{p}\right) E(u_n) - \lambda \left(\frac{1}{q} - \frac{1}{p}\right) B(u_n) = I_\lambda(u_n) - \frac{1}{p} I'_\lambda(u_n) u_n \le c + o(1)\|u_n\| \tag{4.7}$$

for some constant c. Assume that $\|u_n\| \to \infty$ and set $v_n = \frac{u_n}{\|u_n\|}$. Then we may assume that $v_n \rightharpoonup v$ in X and $v_n \to v$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$. From

$$\int_\Omega \nabla u_n \nabla \phi - a(x)|u_n|^{p-2}u_n\phi - \lambda \int_{\partial\Omega} |u_n|^{q-2}u_n\phi = o(1)\|\phi\|, \quad \forall \phi \in X \tag{4.8}$$

we get, dividing it by $\|u_n\|^{p-1}$,

$$\int_\Omega a(x)|v_n|^{p-2}v_n\phi \to 0 \quad \forall \phi \in X$$

so that

$$\int_\Omega a(x)|v|^{p-2}v\phi = 0 \quad \forall \phi \in X.$$

This equality implies that $a|v|^{p-2}v = 0$ a.e. in Ω. Hence $av \equiv 0$. Taking now $\phi = v$ in (4.8), we obtain

$$\int_\Omega \nabla v_n \nabla v - \lambda \|u_n\|^{q-2} \int_{\partial\Omega} |v_n|^{q-2}v_n v \to 0.$$

Thus

$$\int_\Omega \nabla v_n \nabla v \to 0$$

and since $v_n \rightharpoonup v$ in X, we get $\int_\Omega |\nabla v|^2 = 0$. So v must be a constant. From $av \equiv 0$, we deduce that $v \equiv 0$. Finally, from (4.7), dividing it by $\|u_n\|^2$ we obtain $E(v_n) \to 0$. Therefore, by Lemma 4.1, we have $v_n \to 0$ in X, which contradicts $\|v_n\| = 1$.

So (u_n) must be bounded, and up to a subsequence, $u_n \rightharpoonup u$ in X and $u_n \to u$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$. Taking $\phi = u_n - u$ in (4.8) we get

$$\int_\Omega |\nabla u_n|^2 \to \int_\Omega |\nabla u|^2$$

and consequently $\|u_n\|^2 \to \|u\|^2$. By the uniform convexity of X, we infer that $u_n \to u$ in X. □

We prove now a multiplicity result for positive solutions of (P_λ) for $\lambda \in (0, \overline{\lambda})$. First of all, by Proposition 4.5 or Proposition 4.8, we know that $\overline{\lambda} \geq \lambda_0 > 0$. We proceed now as in [9] to obtain a solution by the variational form of the sub-supersolution method. A version of this method for a problem with Neumann boundary conditions has been proved in [11, Theorem 3]. We shall use a slightly different version of this result, namely:

Theorem 4.14. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ be Carathéodory functions such that for every $R > 0$ there exists $M = M(R) > 0$ satisfying $|f(x,s)| \leq M$ if $(x,s) \in \Omega \times [-R,R]$ and $|g(x,s)| \leq M$ if $(x,s) \in \partial\Omega \times [-R,R]$. If $\underline{u}, \overline{u} \in H^1(\Omega) \cap L^\infty(\Omega) \cap L^\infty(\partial\Omega)$ are a weak subsolution and supersolution of (P_λ), respectively, and $\underline{u} \leq \overline{u}$ a.e. in Ω then (P_λ) has a solution u satisfying*

$$I_\lambda(u) = \min\{I_\lambda(v) : \ v \in H^1(\Omega), \ \underline{u} \leq v \leq \overline{u} \text{ a.e. in } \Omega\}.$$

The proof of this result can be carried out following the proof of [11, Theorem 3]. As a matter of fact, the functional I_λ is not coercive but still bounded from below on the set

$$M := \{u \in H^1(\Omega) : \ \underline{u} \leq u \leq \overline{u} \ \text{ a.e. in } \Omega\}.$$

Let us pick $0 < \mu < \overline{\lambda}$ and prove that (P_μ) has two positive solutions. From the definition of $\overline{\lambda}$ we can take $\mu' \in (\mu, \overline{\lambda}]$ such that $(P_{\mu'})$ has a positive solution $u_{\mu'}$. Now, we make good use of the positive eigenfunction ϕ_1 associated to the smallest eigenvalue σ_1 of (2.1) to build up a suitable positive weak subsolution. We consider the smallest eigenvalue $\hat{\sigma}_1 := \sigma_1(\mu) < 0$ of (2.1) and the corresponding positive eigenfunction $\hat{\phi}_1 = \phi_1(\mu)$. Then $\varepsilon\hat{\phi}_1$ is a strict weak subsolution of (P_μ) if $\varepsilon > 0$ is sufficiently small. Moreover, we can choose $\varepsilon > 0$ such that $\varepsilon\hat{\phi}_1 \le u_{\mu'}$. By Theorem 4.14 with $\underline{u} = \varepsilon\hat{\phi}_1$ and $\overline{u} = u_{\mu'}$, we obtain a solution u_0 of (P_μ) such that

$$I_\mu(u_0) = \min\{I_\mu(v) : \ v \in H^1(\Omega), \ \varepsilon\hat{\phi}_1 \le v \le u_{\mu'} \text{ a.e. in } \Omega\}.$$

In particular, $u_0 > 0$ in $\overline{\Omega}$. Moreover, by the strong maximum principle and the boundary point lemma we have $\varepsilon\hat{\phi}_1 < u_0 < u_{\mu'}$ on $\overline{\Omega}$. It follows that u_0 is a local minimizer of I_μ with respect to the $C^1(\overline{\Omega})$ topology. We may then argue as in [10, Lemma 6.4] to deduce that u_0 is a local minimizer of I_μ with respect to the $H^1(\Omega)$ topology. Now we use an argument from [9]: let $\delta > 0$ such that u_0 minimizes I_μ in $B(u_0, \delta)$ and $0 \notin B(u_0, \delta)$. If u_0 is not a strict minimizer then there exists $v_0 \in B(u_0, \delta)$, $v_0 \not\equiv 0$ such that $I_\mu(v_0) = I_\mu(u_0)$, in which case v_0 is also a local minimizer of I_μ, and consequently a solution of (P_μ). Now, if u_0 is a strict minimizer then, by [8, Theorem 5.10], we infer that for $r > 0$ sufficiently small we have

$$I_\mu(u_0) < \inf\{I_\mu(u) : \ u \in H^1(\Omega), \|u - u_0\| = r\},$$

so that I_μ has the mountain-pass geometry (note that if $w \in A^+$ then $I_\mu(tw) \to -\infty$ as $t \to \infty$). Finally, by Proposition 4.13, I_μ satisfies the Palais-Smale condition, and since I_μ is even the mountain-pass theorem provides a second positive solution of (P_μ).

5. Unbounded subcontinuum

In this section we assume (1.8) and that a changes sign. Moreover, we assume $p < \frac{2N}{N-2}$ if $N > 2$. According to a bifurcation argument developed in [17, 19] we discuss the existence of a global subcontinuum of positive solutions bifurcating from the trivial line $\{(\lambda, 0)\}$. Note that in view of the condition $q < 2$ the nonlinearity in (P_λ) is not differentiable at $u = 0$, so that we can not apply the standard local bifurcation theory [7] directly. To overcome this difficulty we investigate the existence of a global subcontinuum of positive solutions for a regularized version of (P_λ). The regularized problem is formulated as

$$\begin{cases} -\Delta u = a(x)u^{p-1} & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = \lambda|u + \epsilon|^{q-2}u & \text{on } \partial\Omega, \end{cases} \tag{$Q_{\lambda,\epsilon}$}$$

where $\epsilon > 0$. Indeed, the mapping $t \mapsto |t + \epsilon|^{q-2}t$ is smooth at $t = 0$. We remark that $(Q_{\lambda,0}) = (P_\lambda)$, which means that (P_λ) is the limiting case of $(Q_{\lambda,\epsilon})$ as $\epsilon \to 0^+$. To study the existence of bifurcation points on the trivial line $\{(\lambda, 0)\}$ for $(Q_{\lambda,\epsilon})$, we consider the linearized eigenvalue problem at $u = 0$

$$\begin{cases} -\Delta \phi = \sigma\phi & \text{in } \Omega, \\ \frac{\partial \phi}{\partial \mathbf{n}} = \lambda\epsilon^{q-2}\phi & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

This problem has a unique principal eigenvalue σ_1, which is simple. Moreover we see that $\sigma_1 > 0$ for $\lambda < 0$, $\sigma_1 = 0$ for $\lambda = 0$, and $\sigma_1 < 0$ for $\lambda > 0$. If we denote by ϕ_1 a corresponding positive eigenfunction to σ_1 then ϕ_1 is a positive constant when $\lambda = 0$.

Now we can prove the following result for $(Q_{\lambda,\epsilon})$:

Proposition 5.1. *Let $p < \frac{2N}{N-2}$ if $N > 2$, and $\epsilon > 0$. Assume (1.8) and that a changes sign. Then the following assertions hold:*

(1) If u_n is a positive solution of $(Q_{\lambda,\epsilon})$ for $\lambda = \lambda_n$ such that $\lambda_n \to \lambda^*$ for some $\lambda^* \in \mathbb{R}$ and $u_n \to 0$ in $C(\overline{\Omega})$ then $\lambda^* = 0$.

(2) There exists $\Lambda_\epsilon > 0$ such that $(Q_{\lambda,\epsilon})$ has no positive solutions for $\lambda \ge \Lambda_\epsilon$.

(3) The set of positive solutions of $(Q_{\lambda,\epsilon})$ around $(\lambda, u) = (0,0)$ consists of a curve $(\lambda, u) = (\lambda(s), s(1+w(s)))$ parametrized by $s \in (0, \delta_0)$, for some $\delta_0 > 0$. In addition, $\lambda(\cdot) : [0, \delta_0) \to \mathbb{R}$ and $w(\cdot) : [0, \delta_0) \to Z = \{u \in C^{2+\alpha}(\overline{\Omega}) : \int_\Omega u = 0\}$ are continuous and satisfy $\lambda(0) = 0$, $\lambda(s) > 0$ for $s > 0$, and $w(0) = 0$. Thus bifurcation of positive solutions of $(Q_{\lambda,\epsilon})$ at $(0,0)$ to the region $\lambda > 0$ does occur.

(4) $(Q_{\lambda,\epsilon})$ has no positive solutions for $\lambda = 0$ within a neighborhood of $u = 0$ in $C(\overline{\Omega})$.

(5) The curve $(\lambda(s), s(1 + w(s)))$, $s \in [0, \delta_0)$, can be extended as a positive solution subcontinuum of $(Q_{\lambda,\epsilon})$, denoted by $\mathcal{C}_\epsilon$, so that it is unbounded in $(-\infty, \Lambda_\epsilon) \times C(\overline{\Omega})$.

Remarks on further results with $(Q_{\lambda,\epsilon})$ for $\epsilon \geq 0$ are given as follows.

Remark 5.2.

(1) Assume that an *a priori* upper bound for positive solutions for $(Q_{\lambda,\epsilon})$ exists for every $\epsilon > 0$, i.e. for any $\mu > 0$ there exists a constant $C_\epsilon > 0$ such that for any positive solution u of $(Q_{\lambda,\epsilon})$ with $|\lambda| \leq \mu$ we have

$$\|u\|_{C(\overline{\Omega})} \leq C_\epsilon, \tag{5.2}$$

Then assertions (1), (2) and (4) of Proposition 5.1 ensure that $\{\lambda \in \mathbb{R} : (\lambda, u) \in \mathcal{C}_\epsilon\} = (-\infty, \overline{\lambda}_\epsilon]$ for some $\overline{\lambda}_\epsilon \in (0, \Lambda_\epsilon]$. The inequality (5.2) is still an open question. We refer to [10] for *a priori* upper bounds for positive solutions of (1.4).

(2) Assertions (1), (2) and (4) in Proposition 5.1 are valid for (P_λ). Assume that (5.2) holds for $\epsilon = 0$, and moreover, C_ϵ is provided uniformly for $\epsilon \geq 0$. Then, by the topological analysis proposed by Whyburn [22, Theorem 9.1], we can deduce from Proposition 5.1 that (P_λ) has a unbounded subcontinuum $\mathcal{C}_0$ of positive solutions, bifurcating to the region $\lambda > 0$ at $(0,0)$ and satisfying $\{\lambda \in \mathbb{R} : (\lambda, u) \in \mathcal{C}_0\} = (-\infty, \overline{\lambda}]$ as described in Figure 5. This is achieved by considering the limiting behavior of $\mathcal{C}_\epsilon$ as $\epsilon \to 0^+$.

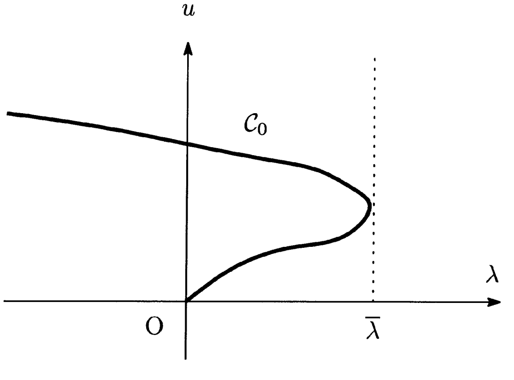

FIGURE 5. A unbounded subcontinuum of positive solutions of (P_λ) when the uniform *a priori* upper bound (5.2) with respect to $\epsilon \geq 0$ is assumed.

The proofs for the results mentioned in this section are to appear somewhere else.

REFERENCES

[1] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, SIAM Rev. **18**, (1976), 620–709.
[2] A. Ambrosetti, H. Brezis, and G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems, J. Funct. Anal. **122**, (1994), 519–543.
[3] C. Bandle, A.M. Pozio, A. Tesei, Existence and uniqueness of solutions of nonlinear Neumann problems, Math. Z. **199**, (1988), 257–278.
[4] K. J. Brown, The Nehari manifold for a semilinear elliptic equation involving a sublinear term, Calc. Var. Partial Differential Equations **22**, (2005), 483–494.
[5] K. J. Brown and T.-F. Wu, A fibering map approach to a semilinear elliptic boundary value problem, Electron. J. Differential Equations **2007**, No. 69, (2007), 9pp.
[6] K. J. Brown and Y. Zhang, The Nehari manifold for a semilinear elliptic equation with a sign-changing weight function, J. Differential Equations **193**, (2003), 481–499.
[7] M. G. Crandall and P. H. Rabinowitz, Bifurcation, perturbation of simple eigenvalues and linearized stability, Arch. Rational Mech. Anal. **52**, (1973), 161–180.
[8] D. G. de Figueiredo, Lectures on the Ekeland Variational Principle with Applications and Detours. Tata Inst. Fund. Res. Lectures Math. Phys. 81, Springer (1989)
[9] D. G. de Figueiredo, J.-P. Gossez, P. Ubilla, Multiplicity results for a family of semilinear elliptic problems under local superlinearity and sublinearity, J. Eur. Math. Soc. (JEMS) **8**, (2006), 269–286.
[10] J. Garcia-Azorero, I. Peral, and J. D. Rossi, A convex-concave problem with a nonlinear boundary condition, J. Differential Equations **198**, (2004), 91–128.
[11] J. García-Melián, J. D. Rossi, and J. C. Sabina de Lis, Limit cases in an elliptic problem with a parameter-dependent boundary condition, Asympt. Anal. **73**, (2011), 147–168.
[12] D. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order, Second edition, Springer-Verlag, Berlin, 1983.
[13] C. Morales-Rodrigo and A. Suárez, Uniqueness of solution for elliptic problems with non-linear boundary conditions, Comm. Appl. Nonlinear Anal. **13**, (2006), 69–78.
[14] M. H. Protter and H. F. Weinberger, Maximum principles in differential equations, Springer-Verlag, New York, 1984.
[15] H. Ramos Quoirin and K. Umezu, The effects of indefinite nonlinear boundary conditions on the structure of the positive solutions set of a logistic equation, J. Differential Equations **257**, (2014), 3935–3977.
[16] H. Ramos Quoirin and K. Umezu, Positive steady states of an indefinite equation with a nonlinear boundary condition: existence, multiplicity and asymptotic profiles, preprint.
[17] H. Ramos Quoirin and K. Umezu, Bifurcation for a logistic elliptic equation with nonlinear boundary conditions: A limiting case, J. Math. Anal. Appl. **428**, (2015), 1265–1285.
[18] H. Ramos Quoirin and K. Umezu, On a concave-convex elliptic problem with a nonlinear boundary condition, Ann. Mat. Pura Appl. (4), (2015), online published.
[19] H. Ramos Quoirin and K. Umezu, An indefinite concave-convex equation under a Neumann boundary condition I, preprint.
[20] J. D. Rossi, Elliptic problems with nonlinear boundary conditions and the Sobolev trace theorem, Stationary partial differential equations, Vol.II, 311–406, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2005.
[21] N. Tarfulea, Existence of positive solutions of some nonlinear Neumann problems, An. Univ. Craiova Ser. Mat. Inform. **23**, (1998), 9–18.
[22] G. T. Whyburn, Topological analysis, Second, revised edition, Princeton Mathematical Series, No. 23, Princeton University Press, Princeton, N.J., 1964.
[23] T.-F. Wu, A semilinear elliptic problem involving nonlinear boundary condition and sign-changing potential, Electron. J. Differential Equations **2006**, No. 131, (2006), 15 pp.

H. RAMOS QUOIRIN
UNIVERSIDAD DE SANTIAGO DE CHILE, CASILLA 307, CORREO 2, SANTIAGO, CHILE
E-mail address: humberto.ramos@usach.cl

K. UMEZU
DEPARTMENT OF MATHEMATICS, FACULTY OF EDUCATION, IBARAKI UNIVERSITY, MITO 310-8512, JAPAN
E-mail address: kenichiro.umezu.math@vc.ibaraki.ac.jp